process has been developed to prevent self-fertilization, indicates that unlikeness instead of favoring fertilization is a hindrance.

So valuable have been the evolutionary advantages of sexual reproduction in increasing variability that many contrivances have been perfected to insure the fulfilment of the function responsible for its creation. Self-sterility or self-impotency is one of the many special adaptations which serve this purpose. The evidence for selective fertilization favoring organisms of the same type, or self-prepotency, is limited just now to one or possibly two species. Will it not be strange to find it so restricted? Is it not more likely to be a general phenomenon manifested in some degree by many organisms? Even in those cases where cross-fertilization is made imperative by a physiological impediment to self-fertilization the same tendency may operate although overwhelmed by the special adaptation. One cannot insist that such is the case, with the evidence isolated as it is at present. Neither is it maintained that the reaction of the cytoplasm of the pollen tubes with the tissues of the host, preceding fertilization, has any relation with the processes which go on within the cells after fertilization. But the prepotency of germ cells acting upon the same or similar individuals which produced them is another indication that homogeneity, likeness, similarity, familiarity, or however it may be described, in protoplasmic structure is consistent with and favorable to the highest developmental efficiency.

GROUPS GENERATED BY TWO OPERATORS, s_1, s_2, *WHICH SATISFY THE CONDITIONS* $s_1^m = s_2^n$, $(s_1s_2)^k = 1$, $s_1s_2 = s_2s_1$

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W. R. Hamilton observed, in 1856, that the groups of movements of the five Platonic solids may be defined by means of equations of the form

$$s_1^m = s_2^n = (s_1s_2)^k = 1.$$

Various generalizations of these groups were obtained during recent years by means of equations of the form

$$s_1^m = s_2^n,\ (s_1s_2)^k = 1.^{*}$$

In both of these cases only a few special values of m, n, k were considered. On the contrary, general values of m, n, k are considered in the present note, but an additional condition is imposed on s_1, s_2; viz., the condition that they shall be commutative operators. Hence all the groups generated by these two operators are abelian.

The main result obtained in this note may be stated as follows: *If two commutative operators,* s_1 *and* s_2, *satisfy the conditions* $s_1^m = s_2^n$, $(s_1s_2)^k = 1$

but are not restricted otherwise they generate the direct product of two cyclic groups of orders $(m + n)k/d$ and d, respectively, d being the highest common factor of m, n, k. In particular, a necessary and sufficient condition that the group G generated by s_1 and s_2 is cyclic is that m, n, k have no common factor greater than unity. For instance, the cyclic group of order 75 is determined by two commutative operators which satisfy the equations

$$s_1^6 = s_2^9,\ (s_1s_2)^5 = 1.$$

This group is also determined by $s_1^{10} = s_2^{15}$, $(s_1s_2)^3 = 1$, etc.

It may first be noted that G is always cyclic when m and n are relatively prime. In fact, when this condition is satisfied and p is any prime number which divides m then the Sylow subgroup of order p^α contained in the group generated by s_1 includes the Sylow subgroup of order p^β contained in the group generated by s_2, and $\alpha > \beta$ when k is divisible by p. When k is not divisible by p then $\alpha = \beta = 0$. Hence it results that the Sylow subgroups of G whose orders are not prime to both of the numbers m and n are cyclic whenever m and n are relatively prime. The other Sylow subgroups of G must also be cyclic since they are generated by s_1^m. This proves that G is cyclic whenever m and n are relatively prime.

When m and n are not relatively prime let d_1 represent their highest common factor. Just as before it follows that the Sylow subgroups of G whose orders are prime to d_1 are cyclic, and those whose orders are not prime to d_1 must also be cyclic when k is prime to d_1 since the generator of such a Sylow subgroup in the group generated by s_1 must be the inverse of such a subgroup in the group generated by s_2. It has, therefore, been proved that a sufficient condition that G be cyclic is that the three numbers m, n, k have no common factor greater than unity.

To prove that the order of G is $(m + n)k$ when this condition is satisfied it is only necessary to prove that the orders of s_1 and s_2 divide this number and that G contains at least one operator whose order is equal to this number. The former of these facts results directly from the following equations:

$$s_1^k = s_2^{-k},\ s_1^{km} = s_2^{-km} = s_2^{kn},\ s_2^{k(m+n)} = 1 = s_1^{k(m+n)}$$

To establish the latter fact it may be noted that when p^α, $\alpha > 0$, is the highest power of p which divides k, and p^β is the highest power of p which divides $m + n$, then we let t_1 represent a constituent of s_1 of order $p^{\alpha+\beta}$ and $t_1^{-1+lp^\beta}$ the corresponding constituent of s_2, where l is so chosen that $t_1^n = t_1^{-n+nlp^\beta}$. As n is prime to p, the product nl represents all the residues prime to p mod p^α, and hence it follows that G involves the cyclic subgroup of order $p^{\alpha+\beta}$.

It remains only to prove that when q^γ is the highest power of the prime number q which divides $m + n$, q being prime to k, G involves a cyclic subgroup of order q^γ. Let t_2 be a generator of such a group. As $t_2^{m+n} = 1$ it follows that $t_2^m = (t_2^{-1})^n$, and hence s_1 may be supposed to have

t_2 as its constituent whose order is a power of q while s_2 has t_2^{-1} as its corresponding constituent. It has, therefore, been proved that when m, n, k do not have a highest common factor greater than unity, and s_1 and s_2 are commutative, the largest group which they can generate subject to the condition $s_1^m = s_2^n$, $(s_1s_2)^k = 1$ is the cyclic group of order $(m + n)k$.

When the highest common factor of m, n, k is $d > 1$ the Sylow subgroups of G whose orders are prime to d are seen to be cyclic just as in the preceding case. The order of every operator in such a group must divide $(m + n)k/d$, since $s_1^k = s_2^{-k}$ and hence $s_1^{\frac{km}{d}} = s_2^{-\frac{km}{d}} = s_2^{\frac{nk}{d}}$, $s_1^{-\frac{kn}{d}} = s_1^{\frac{km}{d}}$. Let r be any prime divisor of d and let r^ϵ be the highest power of r which divides d while $r^{\theta + \epsilon}$ is the highest power of r which divides $(m + n)k$. From the preceding proof it follows that G involves an operator t_3 of order r^θ since such an operator generates a Sylow subgroup of the group determined by the equations

$$s_3^m = s_4^n, \ (s_3s_4)^{k/d} = 1.$$

Suppose that s_3 generates t_3. If an operator which generates the Sylow subgroup whose order is a power of r contained in the group generated by s_4 is multiplied by an independent operator of order r^ϵ this product and s_3 satisfy the conditions

$$s_1^m = s_2^n, \ (s_1s_2)^k = 1.$$

Hence it results that the Sylow subgroup of G whose order is a power of r involves two independent operators of orders r^θ and r^ϵ, respectively. To complete a proof of the theorem under consideration it is, therefore, only necessary to establish the fact that $r^{\theta + \epsilon}$ is the order of a Sylow subgroup of G.

When k is not divisible by $r^{\epsilon + 1}$ this fact is evident since the order of neither of the two operators s_1, s_2 is divisible by $r^{\theta+1}$ and the order of at least one of them is divisible by r^θ. If the order of the Sylow subgroup in question would exceed $r^{\theta + \epsilon}$ the order of the product of these two operators would have to be divisible by $r^{\epsilon + 1}$. When k is divisible by $r^{\epsilon + 1}$ at least one of the two numbers m, n is not divisible by $r^{\epsilon + 1}$. Suppose that m is not divisible by $r^{\epsilon + 1}$. Hence the order of the Sylow subgroup in question cannot exceed r^ϵ times the order of the corresponding Sylow subgroup in the group generated by s_2. This completes a proof of the theorem in italics announced in the second paragraph of this note.

This theorem is useful in the study of the generalized groups of the regular polyhedrons as may be seen from the fact that if two commutative operators satisfy the equations $s_1^3 = s_2^4$, $(s_1s_2)^2 = 1$, they generate the cyclic group of order 14 or one of its subgroups. This explains why the largest non-abelian group, which is generated by two operators satisfying these conditions, is a direct product of the group of order 7 and some other group (Miller, Blichfeldt, Dickson, *Finite Groups*, 1916, p. 157).

Mr. W. E. Edington, graduate student in the University of Illinois, directed my attention to the desirability of a study of the given general equations as regards commutative operators with a view to simplifying the treatment of the generalized polyhedron groups.

MODEL OF THE LINKAGE SYSTEM OF ELEVEN SECOND CHROMOSOME GENES OF DROSOPHILA

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In previous papers[1] reasons have been given for thinking the spatial relations of genes to be three-dimensional rather than linear as assumed in the hypothesis of Morgan and his colleagues. In particular it has been shown that in models made to represent the spatial relations of the genes in the sex chromosomes of two different species of Drosophila, using the original unmodified data, as reported by Morgan and Bridges[2] and by Metz,[3] the arrangement is apparently three-dimensional. Recently data have become available for modeling in a similar way the interrelations of the genes in the so-called "second chromosome" of Drosophila ampelophila.[4] Two views of such a model are seen in figures 1 and 2. The model shows the linkage relations of eleven genes, those for which the data are most complete and reliable according to Bridges and Morgan. The data are taken from table 140 of the publication mentioned. Each gene is represented in the model by a small ring of wire and it is connected with another gene by a wire as long in inches as is the cross-over value, in per cent, between the two genes joined by the wire. In the figures the model is suspended by S, the view shown in figure 2 being taken at right angles to that shown in figure 1. The eleven genes represented in the model are as follows:

S, star (0.0)	*vg*, vestigial (65.0)
Sk, streak (15.4)	*c*, curved (73.5)
d, dachs (29.0)	*px*, plexus (96.2)
b, black (46.5)	*a*, arc (98.4)
pr, purple (52.7)	*sp*, speck (105.1)
	mr, morula (106.3)

The numbers in parentheses indicate the position of each gene in the linear "map" of Bridges and Morgan (p. 127). It will be observed that seven of the genes in this list are represented in the "map" at distances greater than 50 from star, which lies at 0. In the previous publications already cited, I have maintained that since cross-over percentages of 50 or more have not in any case been observed and are logically impossible as a result of linkage, map distances should not be shown in